This week

1. Section 12.1: coordinate systems
2. Section 12.2: vectors
Coordinate systems in $\mathbb{R}^2$

- Choose an **origin**.
- Choose an **$x$-axis**. The arrow head indicates the positive part.
- The **$y$-axis** is now fixed:
  - The $y$-axis is perpendicular on the $x$-axis.
  - The positive $y$-axis is obtained from the positive $x$-axis by rotating it $90^\circ$ to the left.

Cartesian coordinates in $\mathbb{R}^2$

- Let $P$ be a point in the plane.
- Find the projections $p_1$ and $p_2$ of $P$ on the $x$-axis and $y$-axis.
- The Cartesian coordinates of $P$ are $p_1$ and $p_2$, notation:
  $$P = (p_1, p_2).$$
- Every $P \in \mathbb{R}^2$ corresponds to a **unique** pair of Cartesian coordinates.
Distance in $\mathbb{R}^2$

The distance between two points $P_1$ and $P_2$ with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
1.5 Coordinates in $\mathbb{R}^3$

- In $\mathbb{R}^3$ points have three coordinates: $P = (p_1, p_2, p_3)$.
- The projection of $P$ on the $xy$-plane is $P' = (p_1, p_2, 0)$.
- Regard $P$ as the vertex of a rectangular block.

1.6 Distances in $\mathbb{R}^3$

**Definition**

The distance between two points $P_1$ and $P_2$ with coordinates $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$
Example

Find the distance between \( P_1 = (2, 1, 5) \) and \( P_2 = (-2, 3, 0) \).

Circles in the plane

Definition

The circle with centre \((a, b)\) and radius \(r\) is defined as the set of all points \((x, y)\) that have distance \(r\) to \((a, b)\).

- The equation of the circle with center \((a, b)\) and radius \(r\) is
  \[
  \sqrt{(x - a)^2 + (y - b)^2} = r.
  \]

- This is equivalent to
  \[
  (x - a)^2 + (y - b)^2 = r^2
  \]

- The disk with center \((a, b)\) and radius \(r\) is defined by the equation
  \[
  (x - a)^2 + (y - b)^2 \leq r^2
  \]
Spheres and balls

**Definition**

- **The sphere with centre** \((a, b, c)\) **and radius** \(r\) **is defined as the set of all points** \((x, y, z)\) **that have distance** \(r\) **to** \((a, b, c)\).

- **The ball with centre** \((a, b, c)\) **and radius** \(r\) **is defined as the set of all points** \((x, y, z)\) **for which the distance to** \((a, b, c)\) **is less than or equal to** \(r\).

- The sphere with centre \((a, b, c)\) and radius \(r\) is given by the equation
  
  \[
  (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2
  \]

- The ball with centre \((a, b, c)\) and radius \(r\) is given by the inequality
  
  \[
  (x - a)^2 + (y - b)^2 + (z - c)^2 \leq r^2
  \]

**Vectors**

- A vector is a geometric object with a magnitude and a direction.
- A vector can be regarded as a directed line segment, and can be drawn as an arrow.
- Every vector has an initial- and a terminal point.
- The vector with initial point \(P\) and terminal point \(Q\) is denoted as \(PQ\).
- The name of a vector is bold in printed text, or underlined in handwritten texts: \(\mathbf{v}\).
Length

**Definition**

The length of the vector $\overrightarrow{PQ}$ is defined as the distance between $P$ and $Q$.

- If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ then the length of $\mathbf{v} = \overrightarrow{PQ}$ is $|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ then the length of $\mathbf{v} = \overrightarrow{PQ}$ is $|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

**Example**

Find the length of the vector $\overrightarrow{PQ}$ where $P = (-3, 4, 1)$ and $Q = (-5, 2, 2)$.

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Equality of vectors

**Definition**

Two vectors are equal if they have the same length and the same direction.

**Theorem**

Two vectors $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ are equal if $PQQ'P'$ are the vertices of a parallelogram.
Definition

Een vector is said to be in standard position if the origin is the initial point of the vector.

Theorem

For every vector $\mathbf{v}$ there is exactly one point $A$ such that $\mathbf{v} = \overrightarrow{OA}$.

Definition

The component form of a vector $\mathbf{v} = \overrightarrow{OA}$ is the sequence $\langle v_1, \ldots, v_n \rangle$, where $v_1, \ldots, v_n$ are the coordinates of $A$.

Theorem

Let $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$, then

$$\overrightarrow{PQ} = (q_1 - p_1, \ldots, q_n - p_n).$$

Proof for $n = 2$: see Assisted Self-tuition.
**Theorem**

Let \( \langle u_1, \ldots, u_n \rangle \) be the component form of a vector \( u \). Then the length of \( u \) is

\[
|u| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.
\]

**Proof:**

- Put \( u \) in standard position: \( u = \overrightarrow{OP} \) where \( P \) is a point with coordinates \((u_1, \ldots, u_n)\).
- Then

\[
|u| = |OP| = \sqrt{(u_1 - 0)^2 + \cdots + (u_n - 0)^2} = \sqrt{u_1^2 + \cdots + u_n^2}.
\]

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**Definition**

The zero vector, denoted as \( 0 \), is the vector \( \overrightarrow{PP} \), where \( P \) is an arbitrary point.

- The zero vector has no direction and is therefore denoted as a point.
- The component form of the zero vector is \( \overrightarrow{OO} = \langle 0, \ldots, 0 \rangle \).
- The zero vector has length zero: \(|0| = \sqrt{0^2 + \cdots + 0^2} = 0\).
- The zero vector is the only vector with length 0.
The inverse of a vector

**Definition**

The inverse of \( \mathbf{v} = \overrightarrow{PQ} \) is the vector \( -\mathbf{v} = \overrightarrow{QP} \).

![Diagram showing vector \( \mathbf{v} \) and \( -\mathbf{v} \)]

- If \( \mathbf{v} = (v_1, \ldots, v_n) \), then \( -\mathbf{v} = (-v_1, \ldots, -v_n) \).
- The vectors \( \mathbf{v} \) and \( -\mathbf{v} \) have the same length:

\[
|-\mathbf{v}| = \sqrt{(-v_1)^2 + \cdots + (-v_n)^2} = \sqrt{v_1^2 + \cdots + v_n^2} = |\mathbf{v}|.
\]

Scalar multiplication

**Definition**

Let \( \mathbf{v} \) be a vector and let \( \alpha \) be a real number. The **scalar product** \( \alpha \mathbf{v} \) is defined as follows:

- If \( \alpha > 0 \), then \( \alpha \mathbf{v} \) has the same direction as \( \mathbf{v} \), but it is \( \alpha \) times as long as \( \mathbf{v} \).
- If \( \alpha = 0 \), then \( \alpha \mathbf{v} = 0 \).
- If \( \alpha < 0 \) then \( \alpha \mathbf{v} \) the inverse of \( |\alpha| \mathbf{v} \), hence \( \alpha \mathbf{v} = -|\alpha| \mathbf{v} \).

**Theorem**

If \( \mathbf{v} = (v_1, \ldots, v_n) \), then \( \alpha \mathbf{v} = (\alpha v_1, \ldots, \alpha v_n) \) for all real numbers \( \alpha \).

- The number \( \alpha \) is called a **scalar**.
Scalar multiplication

Example (in $\mathbb{R}^2$)
Let $v = \overrightarrow{PQ}$ with $P = (3, 2)$ and $Q = (1, -2)$. Find the terminal point of the vector $-\frac{1}{2}v$ with initial point $P$.

Addition

Definition – head-to-tail construction
Let $u$ and $v$ be two vectors. The sum of $u$ and $v$ (notation: $u + v$) is defined as follows:

1. Write $u = \overrightarrow{PQ}$ for two points $P$ and $Q$.
2. Find a point $R$ such that $v = \overrightarrow{QR}$.
3. Define $u + v = \overrightarrow{PR}$. 
**Theorem – component-wise addition**

Let \( \mathbf{u} = (u_1, \ldots, u_n) \) and \( \mathbf{v} = (v_1, \ldots, v_n) \), then the component form of the sum of \( \mathbf{u} \) and \( \mathbf{v} \) is

\[
\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n).
\]

**Proof:** see Assisted Self-tuition

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**Theorem – parallelogram law**

Let \( \mathbf{u} = \overrightarrow{OP} \) and \( \mathbf{v} = \overrightarrow{OQ} \) be two vectors in standard position. Let \( \mathbf{R} \) be the terminal point of the component form of \( \mathbf{u} + \mathbf{v} \), then \( O, P, Q \) and \( R \) are the vertices of a parallelogram.

- Since \( \mathbf{v} = \overrightarrow{PR} \), the line segment \( OQ \) is parallel to \( PR \), and \( OQ \) and \( PR \) have the same length.

**Corollary – commutativity of the addition**

For all vectors \( \mathbf{u} \) and \( \mathbf{v} \) we have \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \).

- Vector \( \overrightarrow{QR} \) has the same length and direction as \( \overrightarrow{OP} = \mathbf{u} \), hence

\[
\mathbf{u} + \mathbf{v} = \overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR} = \mathbf{v} + \mathbf{u}.
\]
3.8

**Properties**

- Let $u$, $v$ and $w$ be vectors, and let $a$ and $b$ be scalars, then
  
  1. $u + v = v + u$
  2. $(u + v) + w = u + (v + w)$
  3. $u + 0 = u$
  4. $u + (-u) = 0$
  5. $0u = 0$
  6. $1u = u$
  7. $a(bu) = (ab)u$
  8. $a(u + v) = au + av$
  9. $(a + b)u = au + bu$

Property 2 is called the **associativity** of the addition, and it justifies the notation $u + v + w$.

- Let $u$ and $v$ be vectors, and let $a$ be a scalar, then
  
  1. $|0| = 0$
  2. If $|u| = 0$ then $u = 0$
  3. $|au| = |a||u|$
  4. $|u + v| \leq |u| + |v|$
Decomposition in $\mathbb{R}^2$

Example

Let $u = \langle 1, 2 \rangle$ and let $v = \langle 3, 2 \rangle$. Decompose $x = \langle 3, 4 \rangle$ along $u$ and $v$.

Orthogonal decomposition in $\mathbb{R}^2$

Definition

Let $x$ be a vector in $\mathbb{R}^2$. The orthogonal decomposition of $x$ is a decomposition along $i$ and $j$.

Theorem

Let $x$ be a non-zero vector in $\mathbb{R}^2$. The orthogonal decomposition of $x$ is

$$x = |x| \cos \theta \, i + |x| \sin \theta \, j,$$

where $\theta$ is the angle that $x$ makes with the positive $x$-axis.
Example

A 75 N weight is suspende by two wires as shown in the picture above. Find the forces $F_1$ and $F_2$ acting in both wires.

Newton’s first law

If an object is at rest, the total sum of the forces exerted on the object is equal to zero.

Example (continued)